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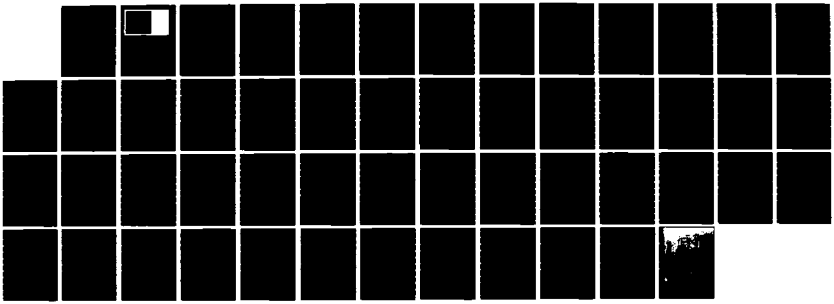

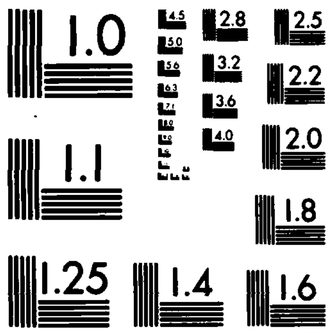

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NUMERICAL ANALYSIS OF BOUNDARY VALUE PROBLEM OF ELLIPTIC TYPE BY MEANS PENALTY AND THE FINITE DIFFERENCE AND ITS APPLICATION TO FREE BOUNDARY PROBLEM

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NUMERICAL ANALYSIS OF BOUNDARY VALUE PROBLEM OF ELLIPTIC TYPE BY MEANS PENALTY AND THE FINITE DIFFERENCE AND ITS APPLICATION TO FREE BOUNDARY PROBLEM

T. Hanada[*], H. Kawarada[**] and O. Pironneau[***]

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ABSTRACT

The authors ~~We~~ study a numerical method for solving free boundary problems of elliptic type. Usually these problems are prescribed with two boundary conditions on the free boundary. One of them is the Dirichlet condition and the other is the Neumann condition. Their ~~Our~~ method is to transform the original problem to an optimization problem. The state equation is approximated by an equation with a penalty term in which the Dirichlet condition on the free boundary is approximately satisfied. The outward normal derivative included in the Neumann condition through the free boundary is calculated by using the asymptotic behavior of the solution of the penalized state equation. ~~Here we present~~ Presented a method to solve this penalized optimization problem. Also the error estimate of the discretized state equation by the finite difference method is given.

AMS (MOS) Subject Classifications: 34E05, 34E99, 35J05, 35J67, 35R35, 39A99, 49-00.

Key Words: Elliptic boundary value problems with discontinuous coefficients, Asymptotic expansion, Penalty methods, Error estimates of finite difference scheme, Optimal control.

Work Unit Number 3 - Numerical Analysis and Scientific Computing

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SIGNIFICANCE AND EXPLANATION

Free boundary problems for elliptic partial differential equations arise in many applications, e.g., phase transition problems including slag flow in the hearth, a wave or jet problem, the equilibrium of plasma, optimal shape design and others. Hence there is interest in the development of efficient and accurate numerical methods for the solution of these problems.

In this report we develop a method for the numerical treatment of these problems. In this method we replace the original problem by an optimization problem in which the state equation contains a "penalty" term. This is particularly useful because the boundary conditions on the free boundary are satisfied approximately; this avoids difficult calculations. Another objective of this report is to give the error estimate of the solution of the discretized state equation by the finite difference method. Our proof uses some of the techniques of the finite element method; it requires a new estimate of the solution of the penalized state equation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

NUMERICAL ANALYSIS OF BOUNDARY VALUE PROBLEM OF ELLIPTIC TYPE BY MEANS PENALTY AND THE FINITE DIFFERENCE AND ITS APPLICATION TO FREE BOUNDARY PROBLEM

T. Hanada*, H. Kawarada** and O. Pironneau***

1. Introduction

This paper is concerned with a numerical method for solving the problem of Dirichlet in an arbitrary domain of R^2 using the finite differences. Consider, for example, the problem of finding ψ^0 such that

$$(1.1) \qquad -\Delta\psi^0 + \lambda_0\psi^0 = f \quad \text{in } \Omega^0, \qquad \psi^0\big|_{\partial\Omega^0} = 0 \ .$$

Let Ω be a rectangle which includes Ω^0 and let χ be the characteristic function of $\Omega-\Omega^0$; then (1.1) may be approximated by

$$(1.2) \qquad -\Delta\psi^\varepsilon + \lambda_0\psi^\varepsilon + \frac{1}{\varepsilon}\chi\psi^\varepsilon = (1-\chi)f \quad \text{in } \Omega\ ,$$

$$\psi^\varepsilon|\partial\Omega = 0 \ .$$

This penalization method was studied in Kawarada (1977), (1979) where it was shown that

$$(1.3) \qquad \psi^\varepsilon\big|_{\partial\Omega^0} = -\sqrt{\varepsilon}\,\frac{\partial\psi^0}{\partial n}\Big|_{\partial\Omega^0} + O(\varepsilon).$$

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When (1.2) is approximated by using, say the 5 points formula for the Laplace operator, then χ must be approximated in such a way as to yield a computable linear system for the values of ψ^{ε} at the grid points. The basic difficulty with such method is that it is not easy to obtain a good approximation of $\partial\psi^{0}/\partial n$ on $\partial\Omega^{0}$ from the discrete approximation of ψ^{ε}.

In this paper we shall show that χ can be approximated (see (2.14)-(2.15)) so as to yield (Theorem 1 and 4).

$$(1.4) \qquad \|\phi_h^{\varepsilon} - \psi^{\varepsilon}\|_{1,\Omega} = O(h^{2/3}) \quad (*)$$

$$(1.5) \qquad \left\| \frac{\partial\psi^0}{\partial n} - \left(-\frac{1}{\sqrt{\varepsilon}}\,\phi_h^{\varepsilon}\right) \right\|_{1/2,\partial\Omega^0} = O(h^{1/3}) ,$$

where h is the mesh size, $\varepsilon=h^{2/3}$, the norms are Sobolev norms, and ϕ_h^{ε} is the linear interpolate from the values of the approximation of ψ^{ε} computed at the grid points.

The proof uses some of the techniques of the finite element method (see Ciarlet (1978) for example); it requires the estimates for $\|\psi^{\varepsilon}-\psi^{0}\|_{\bar{W}^{2,\infty}(\Omega_0)}$ and $\|\psi^{\varepsilon}\|_{\bar{W}^{2,\infty}(\Omega-\Omega_0)}$ which were not known before; thus this paper contributes also to the theory of asymptotic expansion in this respect.

The method of penalization used here (see (1.2)) belongs to the family of artificial domains, capacitance matrices...for which the scientific literature abonds (see for example Lions-Marchuck (1979), Proskurowski and Widlund (1966), (1980) and the bibliography there in); it is quite possible that the techniques used here apply also to these other methods. Finally our practical purpose is to

(*) $\|v\|_{m,G}$ indicates the norm of v in $H^m(G)$.

obtain the solution of a free boundary problem transformed by least squares into an optimum design problem; the method goes as follows:

Let Ω_γ be bounded by a given boundary Γ_0 and a free boundary γ (cf. Fig.1).

We are looking for ϕ and γ such that

(1.6) $$-\Delta\phi + \lambda_0\phi = f \qquad \text{in } \Omega_\gamma$$

(1.7) $$\phi|_\Gamma = 0\ , \qquad \Gamma = \partial\Omega_\gamma = \Gamma_0 \cup \gamma \qquad (\Gamma_0 \cap \gamma = \phi)$$

(1.8) $$\left.\frac{\partial\phi}{\partial n}\right|_\gamma = g\Big|_\gamma$$

where f and g are given in R^2 and λ_0 is a positive constant. n is outward normal to Ω_γ.

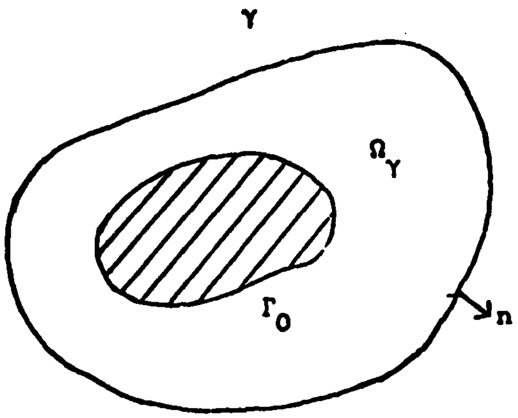

Figure 1

We can now transform this problem into a problem of optimum design, which is an optimal control problem of a distributed parameter system where the control is a part of the boundary.

For a given γ, we define the state equation by

$$-\Delta\psi + \lambda_0\psi = f \quad \text{in } \Omega_\gamma \tag{1.9}$$

$$\psi|_\Gamma = 0 \tag{1.10}$$

which defines $\psi = \psi(x;\gamma)$. Then any solution γ of the problem (1.6)-(1.8) is also solution of

$$\min_{\gamma \in S} \{E(\gamma) = \int_\gamma |\frac{\partial\psi(x,\gamma)}{\partial n} - g|^2 \, ds\} . \tag{1.11}$$

where S is the set of admissible boundaries γ such that $\partial\Omega_\gamma = \Gamma_0 \cup \gamma$ and the state equation has solution in Ω_γ. Problems of the type of (1.11) was successfully studied by Murat-Simon (1977), Chesnais (1975), Pironneau (1976) and Dervieux (1981).

The characteristic of our method to solve (1.11) is summarized as follows:

1° Let Ω be a bounded domain in R^2 such that $\Omega \supset \bar{\Omega}_\gamma$ and $\partial\Omega = \Gamma_0 \cup \Gamma_1$ $(\Gamma_0 \cap \Gamma_1 = \phi)$ for $\forall\gamma \in S$ (cf. Fig.2). We penalize the state equation as follows: For $\varepsilon > 0$,

$$(-\Delta + \lambda_0)\psi^\varepsilon + \frac{1}{\varepsilon}\chi_\gamma\psi^\varepsilon = (1 - \chi_\gamma)f \quad \text{in } \Omega \tag{1.12}$$

$$\psi^\varepsilon|_{\partial\Omega} = 0 \tag{1.13}$$

where χ_γ is the characteristic function of $\Omega - \Omega_\gamma$. If ε is small enough, it is shown that the solution ψ^ε of (1.12) and (1.13) has the following asymptotic behavior on γ:

$$\psi^\varepsilon|_\gamma = -\sqrt{\varepsilon}\,\frac{\partial\psi}{\partial n}\bigg|_\gamma + O(\varepsilon) \quad \text{in a suitable topology}, \tag{1.14}$$

where ψ is the solution of the state equation. This property will be proved in the next section. From (1.14) we observe that the Dirichlet boundary condition (1.13) is approximately satisfied in the error of $O(\sqrt{\varepsilon})$ and also $\left.\frac{\partial\psi}{\partial n}\right|_\gamma$ is approximated by $-\left.\frac{\psi^\varepsilon}{\sqrt{\varepsilon}}\right|_\gamma$ in the same order, which constitutes the key of our method.

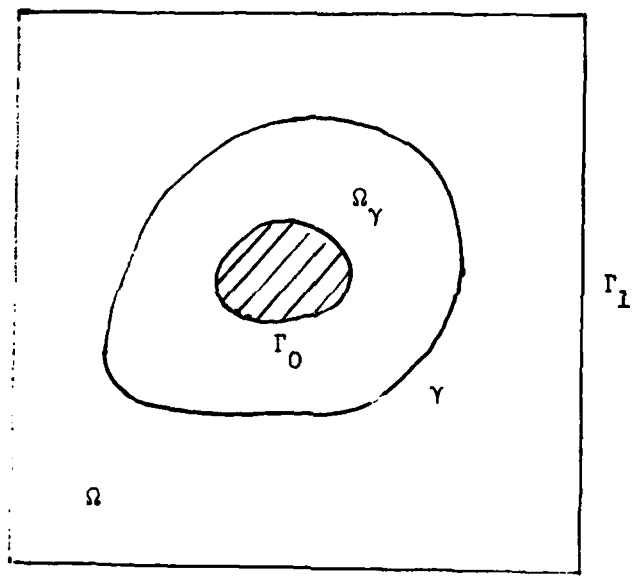

Figure 2

2° By using (1.14), we approximate (1.11) in the following way:

$$\text{(1.15)} \qquad \min_{\gamma \in S} \{E^\varepsilon(\gamma) = \int_\gamma \left|\frac{\psi^\varepsilon}{\sqrt{\varepsilon}} + g\right|^2 ds\} .$$

3° After the penalized state equation (1.12) and (1.13) is discretized by the method of finite difference, (1.15) will be solved by the numerical technique for solving problems of optimum design (Pironneau (1983)).

Finally we would like to mention about the discretization of the penalized state equation in 3°. In order to solve the optimazation problem like (1.11) or (1.15), usually the gradient method is applied together with the finite element method; in which case, the mesh is moved at each iterative approximation of the free boundary. We have frequently experienced that these moving mesh causes serious computational problems.

The plan is following:

1. Introduction
2. The Penalty Method (including main theorems)
3. Proof of Theorem 2
4. Proof of Theorem 3
5. Proof of Theorem 4
6. The algorithm for solving free boundary problem

2. The Penalty Method

The aim of this section is to give the proof of (1.5).

2.1 Let us consider the following problem; given Ω_0 with sufficiently smooth boundary Γ, $(\bar{\Omega}_0 \subset \Omega = (0.1)\times(0.1))$ find ψ^0 such that

(2.1) $$-\Delta\psi^0 + \lambda_0\psi^0 = f \quad \text{in} \quad \Omega_0$$

(2.2) $$\psi^0|_\Gamma = 0 , \qquad \Gamma = \partial\Omega_0 .$$

Then the solution of this problem satisfies

$$\psi^0 \in H^{m+1}(\Omega_0) \cap H_0^1(\Omega_0) \tag{2.3}$$

$$\left.\frac{\partial\psi^0}{\partial n}\right|_\Gamma \in H^{m-\frac{1}{2}}(\Gamma) \tag{2.4}$$

for any $f \in H^{m-1}(\Omega_0)$ $(m \geq 0)$.

2.2 We shall penalize the problem (2.1) and (2.2) as follows; For $\forall\varepsilon > 0$,

$$(-\Delta + \lambda_0)\psi^\varepsilon + \frac{1}{\varepsilon}\chi\,\psi^\varepsilon = (1-\chi)f \qquad \text{in } \Omega \tag{2.5}$$

$$\psi^\varepsilon|_{\partial\Omega} = 0 \tag{2.6}$$

where χ is the characteristic function of $\Omega_1 = \Omega - \bar{\Omega}_0$. It is checked that the solution ψ^ε satisfies

$$\psi^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega) \tag{2.7}$$

$$\psi^\varepsilon \to (1-\chi)\psi^0 \qquad \text{strongly in } H^1(\Omega) \qquad \text{as } \varepsilon \to 0. \tag{2.8}$$

2.3 Let us approximate ψ^ε by the solution ϕ_{ij}^ε of the discrete problem corresponding to (2.5) and (2.6):

$$\{(-\Delta_h + \lambda_0)\phi^\varepsilon\}_{ij} + \frac{1}{\varepsilon}(\chi_h\phi^\varepsilon)_{ij} = ((1-\chi_h)f)_{ij}, \qquad (i,j) \in \overset{\circ}{\Omega}, \tag{2.9}$$

$$(\phi)_{ij} = 0 \qquad (i,j) \in \overset{\sim}{\Omega}, \tag{2.10}$$

$\tilde{\Omega}$ is the set of lattice points which divide Ω into $N \times N$ equidistant meshes ($N = \frac{1}{h}$). $\widetilde{\partial\Omega}$ consists of the lattice points laid on $\partial\Omega$. Δ_h is the five points formula for Δ. In order to define the modified characteristic function χ_h, we need some notations and definitions.

(2.11) $$H_h^1(\Omega) = \{w_h \in C^0(\Omega);\ w_h|_{T_j} \text{ is affine on } T_j\}$$

T_j = triangle obtained by dividing each rectangle of the mesh by its first diagonal (cf. Fig.3).

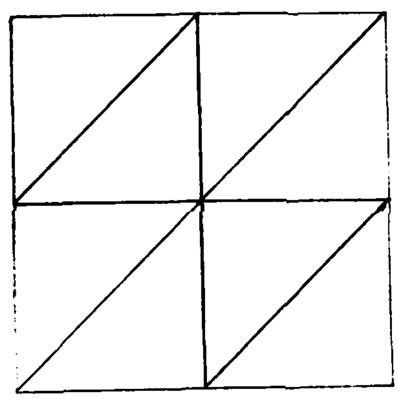

Figure 3

Define $w^{ij}(x)$ by

$w^{ij} \in H_h^1(\Omega)$ and

$w^{ij}(lh, mh) = \delta_{il}\ \delta_{jm}$ for any (l, m).

Further define

(2.12) $$\sigma_h^{ij} = \{x \mid w^{ij}(x) > 0\}$$

(2.13) $\tilde{D}_h = \{(ij) \mid \sigma_h^{ij} \cap \Omega_0 \neq \phi$ and $\sigma_h^{ij} \cap \Omega_1 \neq \phi\}$.

Then

(2.14) $$D_h = \bigcup_{(ij)\in\tilde{D}_h} \sigma_h^{ij}$$

defines a narrow strip along Γ. With the notations above, $\chi_h = \chi_h(x)$ is defined so as to satisfy the following conditions:

i) $\chi_h(x) = 1$ in $\Omega_{1h} = \Omega_1 - \Omega_1 \cap D_h$;

ii) $\chi_h(x) = 0$ in $\Omega_{0h} = \Omega_0 - \Omega_0 \cap D_h$;

iii) $0 \leq \chi_h(x) \leq 1$ in D_h

iv) $\| RS \| = \int_{\Gamma^\perp(s)} \chi_h(x)\, d\Gamma^\perp$ (see Fig. 4, $\|RS\|$ implies the distance between R and S).

The meaning of iv) is clarified in the proof of Theorem 4 stated later. On the other hand, it plays an important role in solving (1.15).

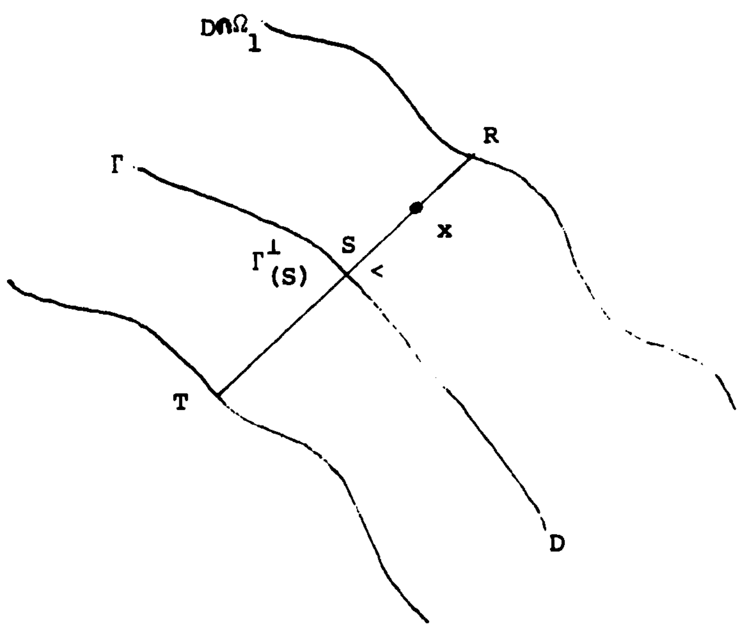

Figure 4

As two examples of χ_h, we give

$$\chi_h(x) = \frac{\ell_1(x)}{\ell_1(x)+\ell_2(x)} \quad \text{in} \quad D_h \tag{2.15}$$

$$\chi_h(x) = \frac{\rho+\ell_2(x)}{\ell_1(x)+\ell_2(x)} \quad \text{in} \quad D_h\ . \tag{2.16}$$

where $\ell_1(x)$ and $\ell_2(x)$ are constant on $\Gamma^{\perp}(S)$ and equal to $\|RS\|$ and $\|TS\|$ respectively, and $\rho = \|x-S\|$.

The linear interpolation of ϕ^{ε}_{ij} $((i,j)\in\tilde{\mathfrak{N}})$ is represented by

$$\phi^{\varepsilon}_h(x) = \sum_{ij\in\tilde{\mathfrak{N}}} \phi^{\varepsilon}_{ij}\cdot w^{ij}(x) \tag{2.17}$$

2.4 Now we shall state the following approximate theorem for $\left.\frac{\partial\psi^0}{\partial n}\right|_{\Gamma_0}$ which plays an essential role in our method:

Theorem 1 Suppose $f\in H^m(\Omega_0)$ $(m \geq 5)$ and $\varepsilon = h^{2/3}$ and let h be small enough.

Then

$$\left\| \frac{\partial\psi^0}{\partial n} + \frac{1}{\sqrt{\varepsilon}}\,\phi^{\varepsilon}_h \right\|_{\frac{1}{2},\Gamma} = O(h^{\frac{1}{3}})\ . \tag{2.18}$$

In order to prove Theorem 1, we have to prepare the following three theorems:

Theorem 2 Suppose $f\in H^m(\Omega_0)$ $(m \geq 0)$ and let ε be small enough.

Then

$$\left\| \psi^{\varepsilon} + \sqrt{\varepsilon}\cdot\frac{\partial\psi^0}{\partial n} \right\|_{m-\frac{1}{2},\Gamma} = O(\varepsilon). \tag{2.19}$$

Theorem 3 Suppose $f \in H^m(\Omega_0)$ $(m \geq 5)$ and let ε be small enough. Then

$$\| \psi^\varepsilon \|_{L_\infty(\Omega)} \leq C_0 \tag{2.20}$$

$$\left\| \frac{\partial^2 \psi^\varepsilon}{\partial x_i \partial x_j} \right\|_{C(\bar{\Omega}_0)} \leq C_1 (1 + \sqrt{\varepsilon}) \tag{2.21}$$

$$\left\| \frac{\partial^2 \psi^\varepsilon}{\partial x_i \partial x_j} \right\|_{C(\bar{\Omega}_1)} \leq \frac{C_2}{\sqrt{\varepsilon}} \ . \tag{2.22}$$

C_i $(i = 0, 1, 2)$ is a positive constant independent of ε.

The regularity properties in Theorem 3 are needed to prove the following theorem.

Theorem 4 Suppose $f \in H^m(\Omega_0)$ $(m \geq 5)$. Then

$$\| \phi_h^\varepsilon - \psi^\varepsilon \|_{1,\Omega} \leq O\left(\frac{h^2}{\varepsilon\sqrt{\varepsilon}} + \frac{h}{\sqrt{\varepsilon}}\right). \qquad (= O(h^{\frac{2}{3}}) \text{ when } \varepsilon = h^{\frac{2}{3}}) \ . \tag{2.23}$$

2.5 Here we give the proof of Theorem 1. From Theorems 2 and 4, we have

$$\left\| \frac{\phi_h^\varepsilon}{\sqrt{\varepsilon}} - \frac{\psi^\varepsilon}{\sqrt{\varepsilon}} \right\|_{\frac{1}{2},\Gamma} = O\left(\frac{h^2}{\varepsilon^2} + \frac{h}{\varepsilon}\right), \tag{2.24}$$

$$\left\| \frac{\psi^\varepsilon}{\sqrt{\varepsilon}} + \frac{\partial \psi^0}{\partial n} \right\|_{\frac{1}{2},\Gamma} = O(\sqrt{\varepsilon}). \tag{2.25}$$

Combining (2.24) and (2.25),

$$\left\| \frac{\phi_h^\varepsilon}{\sqrt{\varepsilon}} + \frac{\partial\psi^0}{\partial n} \right\|_{\frac{1}{2},\Gamma} = O(\sqrt{\varepsilon} + \frac{h}{\varepsilon} + (\frac{h}{\varepsilon})^2). \tag{2.26}$$

Thus if we put $\varepsilon = h^{2/3}$, then

$$\left\| \frac{\phi_h^\varepsilon}{\sqrt{\varepsilon}} + \frac{\partial\psi^0}{\partial n} \right\|_{\frac{1}{2},\Gamma} = O(h^{\frac{1}{3}}). \tag{2.27}$$

■

3. Proof of Theorem 2

3.1 We introduce some operators defined between traces on Γ.

(i) Define the mapping

$$T_f : H^{\frac{1}{2}}(\Gamma) \ni a \mapsto \left.\frac{\partial\psi_a}{\partial n}\right|_\Gamma \in H^{-\frac{1}{2}}(\Gamma) \quad : \tag{3.1}$$

ψ_a is the solution of the problem:

$$-\Delta\psi + \lambda_0\psi = f \qquad \text{in } \Omega_0 \tag{3.2}$$

$$\psi|_\Gamma = a \tag{3.3}$$

where $f \in H^{m-1}(\Omega_0)$.

(ii) Define the mapping

$$R^\varepsilon : H^{-\frac{1}{2}}(\Gamma) \ni b \mapsto \psi_b^\varepsilon\big|_\Gamma \in H^{\frac{1}{2}}(\Gamma) \quad : \tag{3.4}$$

ψ_b is the solution of (3.2) with $f \equiv 0$ and the boundary condition

$$(3.5)\qquad (\sqrt{\varepsilon}\,\frac{\partial\psi}{\partial n} + \psi)\big|_\Gamma = b\ .$$

(iii) Define the mapping

$$(3.6)\qquad S^\varepsilon : H^{\frac{1}{2}}(\Gamma) \ni a \mapsto \frac{\partial\psi_a}{\partial n}\Big|_\Gamma \in H^{-\frac{1}{2}}(\Gamma)\quad :$$

where ψ_a^ε is the solution of the problem:

$$(3.7)\qquad -\varepsilon\Delta\psi + \psi = 0 \qquad \text{in } \Omega_1$$

$$(3.8)\qquad \psi|_\Gamma = a$$

$$(3.9)\qquad \psi|_{\partial\Omega} = 0\ .$$

We denote T_f^m, S_m^ε and R_m^ε by the restriction of T_f, S^ε and R^ε to $H^{m+\frac{1}{2}}(\Gamma)$. But, we abbreviate the suffix m hereafter.

3.2

<u>Lemma 3.1</u> Let a, b be arbitrary in $H^{m+1/2}(\Gamma)$. Then

$$(3.10)\qquad T_f(a) - T_f(b) = T_0(a-b)$$

where $T_0 = T_{f=0}$.

<u>Proof</u> Let ψ_s $(s = a,b)$ be the solution of (3.2) and the boundary condition

$$(3.11)\qquad \psi|_\Gamma = s\ .$$

Put $\Psi = \psi_a - \psi_b$. Ψ satisfies

$$(3.12)\qquad -\Delta\Psi + \lambda_0\Psi = 0 \qquad \text{in } \Omega_0\ ,$$

(3.13) $$\Psi|_{\Gamma} = a - b$$

(3.14) $$\frac{\partial\Psi}{\partial n}\bigg|_{\Gamma} = T_0(a-b).$$

On the other hand,

(3.15) $$\frac{\partial\Psi}{\partial n}\bigg|_{\Gamma} = \frac{\partial\psi_a}{\partial n}\bigg|_{\Gamma} - \frac{\partial\psi_b}{\partial n}\bigg|_{\Gamma} = T_f(a) - T_f(b).$$

From (3.14) and (3.15) follows (3.10). ∎

Here it should be noted that T_0 is linear and T_f is non-linear.

<u>Lemma 3.2</u>

T_f, $(R^{\varepsilon})^{-1}$ and S^{ε} are homeomorphic from $H^{m-1/2}(\Gamma)$ to $H^{m+1/2}(\Gamma)(m \geqq 0)$.

<u>Proof</u>

1° T_f is injective from $H^{m+1/2}(\Gamma)$ into $H^{m-1/2}(\Gamma)$. Indeed, let $a,b \in H^{m+1/2}(\Gamma)$ $(a \neq b)$. Suppose $T_f(a) = T_f(b)$. Then, by (3.10)

$$0 = T_f(a) - T_f(b) = T_0(a-b) \neq 0$$

because of the strong maximum principle under the assumption $\lambda_0 > 0$. This is a contradiction.

2° T_f is surjective from $H^{m+1/2}(\Gamma)$ onto $H^{m-1/2}(\Gamma)$ if $\lambda_0 > 0$. In fact, choose any $b \in H^{m-1/2}(\Gamma)$. Then the following problem:

(3.16) $$-\Delta\psi + \lambda_0\psi = f \quad \text{in } \Omega_0$$

(3.17) $$\frac{\partial\psi}{\partial n}\bigg|_{\Gamma} = b$$

has a unique solution $\psi_b \in H^{m+1}(\Omega_0)$ if $\lambda_0 > 0$, which satisfies

$$(3.18) \qquad \psi_b\big|_\Gamma \in H^{m+\frac{1}{2}}(\Gamma) \quad \text{and} \quad b = T_f(\psi_b\big|_\Gamma).$$

3° It is checked that T_f and $(T_f)^{-1}$ are continuous between $H^{m+1/2}(\Gamma)$ and $H^{m-1/2}(\Gamma)$ (see Agmon, Douglis and Nirenberg (1959)).

4° Summing up 1°, 2° and 3°, we see that T_f is a homeomorphism from $H^{m+1/2}(\Gamma)$ onto $H^{m-1/2}(\Gamma)$.

The repeated use of the above arguments proves that R^ε and S^ε are homeomorphic between $H^{m+1/2}(\Gamma)$ and $H^{m-1/2}(\Gamma)$.

■

3.3 Here we give the estimates of the norm of R^ε and S^ε, which are crucial for the proof of Theorem 2.

<u>Lemma 3.3</u> Let ε be small enough and $m \geq 0$. Then

$$(3.19) \qquad \| R^\varepsilon(a) \|_{m-\frac{1}{2},\Gamma} = O(1) \| a \|_{m-\frac{1}{2},\Gamma}, \text{ for } \forall a \in H^{m-1/2}(\Gamma)$$

$$(3.20) \qquad \| R^\varepsilon(a) \|_{m+\frac{1}{2},\Gamma} = O(\frac{1}{\sqrt{\varepsilon}}) \| a \|_{m-\frac{1}{2},\Gamma}, \text{ for } \forall a \in H^{m-1/2}(\Gamma)$$

$$(3.21) \qquad \| R^\varepsilon(a) - a \|_{m-\frac{1}{2},\Gamma} = O(\sqrt{\varepsilon}) \| a \|_{m+\frac{1}{2},\Gamma}$$

for any $a \in H^{m+\frac{1}{2}}(\Gamma)$.

<u>Proof</u> Using Green's formula in the problem defining R^ε, we have

$$(3.22) \qquad \sqrt{\varepsilon}\int_{\Omega_0} (|\nabla\psi|^2 + \lambda_0|\psi|^2)dx + \int_\Gamma |\psi|^2 ds = \int_\Gamma a\psi ds .$$

From (3.22) it follows

(3.23) $$\| \psi \|_{0,\Gamma} \leq \| a \|_{0,\Gamma}$$

(3.24) $$\sqrt{\varepsilon} \, \| \psi \|_{\frac{1}{2},\Gamma} \leq \| a \|_{-\frac{1}{2},\Gamma} \, .$$

Using the standard technique to raise up the regularity property of the solution of partial differential equation, we obtain (3.19) and (3.20). Rewriting (3.5) with an aid of T_0 and R^ε, we have for any $a \in H^{m+1/2}(\Gamma)$

(3.25)
$$\begin{aligned} \| R^\varepsilon(a) - a \|_{m-\frac{1}{2},\Gamma} &= \sqrt{\varepsilon} \, \| T_0 R^\varepsilon(a) \|_{m-\frac{1}{2},\Gamma} \\ &= O(\sqrt{\varepsilon}) \, \| R^\varepsilon(a) \|_{m+\frac{1}{2},\Gamma} \\ &= O(\sqrt{\varepsilon}) \, \| a \|_{m+\frac{1}{2},\Gamma} \qquad \text{(by 3.19).} \end{aligned}$$

Here we have used the boundedness of T_0 from $H^{m+1/2}(\Gamma)$ to $H^{m-1/2}(\Gamma)$. ∎

<u>Lemma 3.4</u> Let ε be small enough and $m \geq 0$. Then

(3.26) $$\left\| S^\varepsilon(a) + \frac{a}{\sqrt{\varepsilon}} \right\|_{m-\frac{1}{2},\Gamma} = O(\sqrt{\varepsilon}) \, \| a \|_{m+\frac{1}{2},\Gamma} \, ,$$

$$\forall a \in H^{m+\frac{1}{2}}(\Gamma).$$

<u>Proof</u> We prove this lemma in two cases. In the first case, we prove the special case; $\Omega_1 = R_+^2$ $(\Omega_0 = R_-^2)$ by using the fourier transformation. We give the plan of the proof in the general geometry in the second case.

1° Let $\hat{\psi}^{\varepsilon}(x_1, \xi) = \int_{-\infty}^{\infty} e^{2\pi i \xi x_2} \psi^{\varepsilon}(x_1, x_2) dx_2$

and $$\hat{a}(\xi) = \int_{-\infty}^{\infty} e^{2\pi i \xi x_2} a(x_2) dx_2, (a \in H^{m+3/2}(\Gamma)) .$$

Here $\psi^{\varepsilon}(x_1, x_2)$ is the solution of (3.7)-(3.9). Then $\hat{\psi}^{\varepsilon}$ satisfies

$$(3.27) \qquad -\frac{\partial^2 \hat{\psi}^{\varepsilon}}{\partial x_1^2} + \frac{1}{\varepsilon}(1 + 4\pi^2 \cdot |\xi|^2 \varepsilon) \hat{\psi}^{\varepsilon} = 0 \qquad \text{in } R_+^2 ,$$

$$(3.28) \qquad \hat{\psi}^{\varepsilon}|_{x_1=0} = \hat{a} .$$

Solving (3.27) and (3.28), we have

$$(3.29) \qquad \hat{\psi}^{\varepsilon} = \hat{a} \cdot \exp\{-\frac{1}{\sqrt{\varepsilon}}(1 + 4\pi^2 \cdot |\xi|^2 \cdot \varepsilon)^{\frac{1}{2}} x_1\} .$$

from which

$$(3.30) \qquad \left.\frac{\partial \hat{\psi}^{\varepsilon}}{\partial x_1}\right|_{x_1=0} = \widehat{S^{\varepsilon}(a)} = -\frac{1}{\sqrt{\varepsilon}}(1 + 4\pi^2 \cdot |\xi|^2 \cdot \varepsilon)^{\frac{1}{2}} \cdot \hat{a} .$$

We compute

$$(3.31) \qquad \widehat{S^\varepsilon(a)} + \frac{1}{\sqrt{\varepsilon}}\hat{a} = -\frac{\hat{a}}{\sqrt{\varepsilon}} \cdot \frac{4\pi^2|\xi|^2\cdot\varepsilon}{1+(1+4\pi^2|\xi|^2\varepsilon)^{\frac{1}{2}}} \cdot$$

$$(3.32) \qquad \int_{-\infty}^{\infty} (1 + 4\pi^2\cdot|\xi|^2)^{m-\frac{1}{2}} \cdot \left|\widehat{S^\varepsilon(\varepsilon)} + \frac{\hat{a}}{\sqrt{\varepsilon}}\right|^2 d\xi$$

$$\leq O(\varepsilon)\int_{-\infty}^{\infty} |\hat{a}|^2 \cdot \frac{(1 + 4\pi^2\cdot|\xi|^2)^{m+\frac{3}{2}}}{[1+(1+4\pi^2\cdot|\xi|^2\varepsilon)^{\frac{1}{2}}]^2} \cdot d\xi$$

$$\leq O(\varepsilon)\int_{-\infty}^{\infty} |\hat{a}|^2 (1 + 4\pi^2\cdot|\xi|^2)^{m+\frac{3}{2}}\, d\xi$$

which implies

$$(3.33) \qquad \left\| S^\varepsilon(a) + \frac{a}{\sqrt{\varepsilon}} \right\|_{m-\frac{1}{2},\Gamma} = O(\sqrt{\varepsilon})\, \| a \|_{m+\frac{3}{2},\Gamma} \cdot$$

By density argument, we have

$$(3.34) \qquad \left\| S^\varepsilon(a) + \frac{a}{\sqrt{\varepsilon}} \right\|_{m-\frac{1}{2},\Gamma} = O(\sqrt{\varepsilon})\, \| a \|_{m+\frac{1}{2},\Gamma}$$

for $\forall a \in H^{m+\frac{1}{2}}(\Gamma)$.

2° Let us now deal with the <u>general case</u>. The domain Ω_1 is a regular simply connected domain; then there exists a (fixed) regular conformal mapping $w = f(z) = u_1 + iu_2$ $(z = x_1 + ix_2)$ which

maps Ω_1 into R^2_+. As a matter of fact, Γ is mapped into the u_2-axis of w-plane. Then the transformed sulution $\Psi^\varepsilon = \psi^\varepsilon(f^{-1}(w))$ satisfies

$$(3.35) \qquad -\varepsilon\cdot\Delta\Psi^\varepsilon + \left|\frac{dz}{dw}\right|^2 \Psi^\varepsilon = 0 \quad \text{in} \quad R^2_+ ,$$

$$(3.36) \qquad \left.\frac{\partial\Psi^\varepsilon}{\partial u_1}\right|_{u_1=0} = A(u_2) = \left|\frac{dz}{dw}\right| a(f^{-1}(w)).$$

By means of the iterative method proposed in the theory of singular perturbation (see Lions (1973)), ψ^ε is asymptotically developed in the following way:

$$(3.37) \qquad \Psi^\varepsilon = \frac{\Psi^{-1}}{\sqrt{\varepsilon}} + \Psi^0 + \sqrt{\varepsilon}\Psi^1 + \cdots + (\sqrt{\varepsilon})^n \Psi^n + \theta^n .$$

Using (3.37), we conclude (3.26). ∎

3.4 Let us note that another description of the problem (2.5) and (2.6) is following:

$$(3.38) \qquad -\Delta\psi_0 + \lambda_0\psi_0 = f \quad \text{in} \quad \Omega_0 ,$$

$$(3.39) \qquad -\varepsilon\Delta\psi_1 + \psi_1 = 0 \quad \text{in} \quad \Omega_1 ,$$

$$(3.40) \qquad \psi_0 = \psi_1 \quad \text{on} \quad \Gamma ,$$

$$(3.41) \qquad \frac{\partial\psi_0}{\partial n} = \frac{\partial\psi_1}{\partial n} \quad \text{on} \quad \Gamma ,$$

(3.42) $$\psi_1\Big|_{\partial\Omega} = 0 .$$

Then this problem is transformed into the transmission equation: Find $a^\varepsilon \in H^{m+1/2}(\Gamma)$ such that

(3.43) $$T_f(a^\varepsilon) = S^\varepsilon(a^\varepsilon).$$

Of course, the solution a^ε of (3.43) is equal to $\psi^\varepsilon|_\Gamma$.

3.5 Using Lemma 3.1, (3.43) is rewritten in the following way:

(3.44) $$T_0(a^\varepsilon - b) - S^\varepsilon(a^\varepsilon) = -T_f(b), \qquad \text{for } \forall b \in H^{m+\frac{1}{2}}(\Gamma).$$

Recalling (2.8),

(3.45) $$a^\varepsilon = \psi^\varepsilon|_\Gamma \to 0 \qquad \text{strongly in } H^{\frac{1}{2}}(\Gamma) \quad \text{as } \varepsilon \to 0 .$$

Therefore we choose $b = 0$ in (3.44). By (3.26) in Lemma 3.4, we have

(3.46) $$T_0(a^\varepsilon) + \frac{a^\varepsilon}{\sqrt{\varepsilon}} - S_1^\varepsilon(a^\varepsilon) = -T_f(0).$$

or

$$\sqrt{\varepsilon}\cdot T_0(a^\varepsilon) + a^\varepsilon = \sqrt{\varepsilon}S_1^\varepsilon(a^\varepsilon) - \sqrt{\varepsilon}T_f(0)$$

where

(3.47) $$S_1^\varepsilon(a^\varepsilon) = S^\varepsilon(a^\varepsilon) + \frac{a^\varepsilon}{\sqrt{\varepsilon}} .$$

By the definition of R^{ε},

$$(3.48) \qquad a^{\varepsilon} = \sqrt{\varepsilon} R^{\varepsilon} S_1^{\varepsilon}(a^{\varepsilon}) - \sqrt{\varepsilon} R^{\varepsilon} T_f(0).$$

<u>Lemma 3.5</u> Let ε be small enough. Then $\sqrt{\varepsilon} R^{\varepsilon} S_1^{\varepsilon}$ is a contraction mapping from $H^{m+1/2}(\Gamma)$ onto itself.

<u>Proof</u> Recalling the definitions of R^{ε} and S_1^{ε}, we see that $\sqrt{\varepsilon} R^{\varepsilon} S_1^{\varepsilon}$ maps $H^{m+1/2}(\Gamma)$ onto $H^{m+1/2}(\Gamma)$. We compute

$$(3.49) \qquad \sqrt{\varepsilon}\, \| R^{\varepsilon} S_1^{\varepsilon}(a^{\varepsilon}) \|_{m+\frac{1}{2},\Gamma} \le \sqrt{\varepsilon} \cdot O\left(\frac{1}{\sqrt{\varepsilon}}\right) \| S_1^{\varepsilon}(a^{\varepsilon}) \|_{m-\frac{1}{2},\Gamma}$$

(by 3.20)

$$\le O(\sqrt{\varepsilon}) \| a^{\varepsilon} \|_{m+\frac{1}{2},\Gamma} \quad \text{(by 3.26)}$$

Thus (3.49) completes this lemma. ∎

Using this lemma, we can solve (3.43):

$$(3.50) \qquad a^{\varepsilon} = (I - \sqrt{\varepsilon}\, R^{\varepsilon} S_1^{\varepsilon})^{-1} (-\sqrt{\varepsilon}\, R^{\varepsilon} T_f(0)) \quad \text{in} \quad H^{m+\frac{1}{2}}(\Gamma).$$

By (3.21) in Lemma 3.3,

$$(3.51) \qquad a^{\varepsilon} = (I - \sqrt{\varepsilon}\, R^{\varepsilon} S_1^{\varepsilon})^{-1} (-\sqrt{\varepsilon}\, T_f(0) + O(\varepsilon)) \quad \text{in} \quad H^{m+\frac{1}{2}}(\Gamma).$$

which implies

$$a^{\varepsilon} = -\sqrt{\varepsilon}\cdot T_f(0) + O(\varepsilon) \qquad \text{in} \quad H^{m+\frac{1}{2}}(\Gamma). \tag{3.52}$$

Here we note that $T_f(0)$ should satisfy

$$T_f(0) = -\left.\frac{\partial\psi^0}{\partial n}\right|_{\Gamma} \in H^{m+\frac{3}{2}}(\Gamma) .$$

Therefore, we have to assume $f \in H^{m+1}(\Omega_0)$ from which follows $\psi^0 \in H^{m+3}(\Omega_0)$.

4. Proof of Theorem 3

4.1 Assume $f \in H^k(\Omega_0)$ $(k \geq 5)$.

Then

$$\psi^0 \in H^1_0(\Omega_0) \cap H^{k+2}(\Omega_0), \tag{4.1}$$

$$\left.\frac{\partial\psi^0}{\partial n}\right|_{\Gamma} \in H^{k+\frac{1}{2}}(\Gamma) \cap C^{k-1,\delta}(\Gamma) \qquad (0 \leq \delta < 1) \tag{4.2}$$

(see Adams (1975)). Using (2.19) in Theorem 3 and (3.2),

$$\psi^{\varepsilon}|_{\Gamma} = O(\sqrt{\varepsilon}) \qquad \text{in} \quad H^{k+\frac{1}{2}}(\Gamma) \cap C^{k-1,\delta}(\Gamma). \tag{4.3}$$

Using (4.3) and the maximum principle, we obtain

$$\| \psi^{\varepsilon} \|_{C(\bar{\Omega}_1)} \leq O(\sqrt{\varepsilon}) \qquad (k \geq 3). \tag{4.4}$$

We compute

(4.5) $$\left\| \nabla \psi^{\varepsilon} \right\|_{k-\frac{1}{2},\Gamma} \leq \left\| \frac{\partial \psi^{\varepsilon}}{\partial n} \right\|_{k-\frac{1}{2},\Gamma} + \left\| \frac{\partial \psi^{\varepsilon}}{\partial s} \right\|_{k-\frac{1}{2},\Gamma} .$$

By (4.3)

(4.6) $$\left\| \frac{\partial \psi^{\varepsilon}}{\partial s} \right\|_{k-\frac{1}{2},\Gamma} \leq O(\sqrt{\varepsilon}) .$$

From the definition of S^{ε}, we have

$$\left.\frac{\partial \psi^{\varepsilon}}{\partial n}\right|_{\Gamma} = S^{\varepsilon}(\psi^{\varepsilon}|_{\Gamma}) \in H^{k-\frac{1}{2}}(\Gamma) \cap C^{k-2,\delta}(\Gamma)$$

By (3.26) and (2.19),

(4.7) $$\left\| \frac{\partial \psi^{\varepsilon}}{\partial n} \right\|_{k-\frac{1}{2},\Gamma} \leq \frac{1}{\sqrt{\varepsilon}} \| \psi^{\varepsilon} \|_{k+\frac{1}{2},\Gamma} \leq O(1) .$$

Combining (4.6) and (4.7),

(4.8) $$\| \nabla \psi^{\varepsilon} \|_{k-\frac{1}{2},\Gamma} \leq O(1) .$$

Similarly, we have

(4.9) $$\| \nabla \psi^{\varepsilon} \|_{k-\frac{1}{2},\partial\Omega} \leq \left\| \frac{\partial \psi^{\varepsilon}}{\partial n} \right\|_{k-\frac{1}{2},\partial\Omega} \leq O(1)$$

Since (4.4), $\left.\frac{\partial \psi^{\varepsilon}}{\partial n}\right|_{\partial\Omega} \in H^{k-\frac{1}{2}}(\partial\Omega) \cap C^{k-2,\delta}(\partial\Omega)$ and $\psi^{\varepsilon}|_{\partial\Omega} = 0$. Put $\nabla\psi^{\varepsilon} = \Psi^{\varepsilon}$. Then Ψ^{ε} satisfies

(4.10) $$-\varepsilon\Delta\Psi^{\varepsilon} + \Psi^{\varepsilon} = 0 \quad \text{in} \quad \Omega_1.$$

From the maximum principle together with (4.8) and (4.9) it follows

(4.11) $$\|\nabla\psi^{\varepsilon}\|_{C(\bar{\Omega}_1)} \leq O(1).$$

Here we have to assume $k \geq 4$ to obtain good regularity of ψ^{ε}.

Repeating similar argument, we have

(4.12) $$\left\| \frac{\partial^2 \psi^{\varepsilon}}{\partial x_i \partial x_j} \right\|_{C(\bar{\Omega}_1)} \leq O\left(\frac{1}{\sqrt{\varepsilon}}\right) \qquad (k \geq 5).$$

5. Proof of Theorem 4

Here we restrict our proof to the case $\lambda_0 = 0$ without loss of generality.

5.1 We will introduce the following scalar product equivalent to the one induced by $H_0^1(\Omega)$:

(5.1) $$(u, v) = \int_{\Omega} \nabla u \nabla v \, dx + \frac{1}{\varepsilon} \sum_{(i,j)\in\tilde{\Omega}} (\chi_h uv)_{ij} h^2 ,$$

for any $u, v \in H_0^1(\Omega)$

and the associated norm

(5.2) $$\| u \| = \sqrt{(u, u)}$$

Let $H_{oh}(\Omega)=\{w_h \in H_h^1(\Omega) \mid w_h|_{\partial\Omega} = 0\}$. Then, since $H_{oh}(\Omega)$ is a closed subspace in $H_0^1(\Omega)$, there exists one and only one element $\theta_h \in H_{oh}(\Omega)$ which minimize $\|\psi^\varepsilon - \phi_h^\varepsilon - w_h\|$ among $w_h \in H_{oh}^1(\Omega)$.

We denote

(5.3) $$v_h = \psi^\varepsilon - \phi_h^\varepsilon - \theta_h .$$

Obviously, v_h satisfies

(5.4) $$(v_h , w_h) = 0 , \quad \forall w_h \in H_{oh}^1(\Omega)$$

and there holds

(5.5) $$\| \psi^\varepsilon - \phi_h^\varepsilon \|^2 = \| \theta_h \|^2 + \| v_h \|^2 .$$

Therefore we may evaluate both $\| \theta_h \|$ and $\| v_h \|$. It follows from (5.4) that for some C (see Ciarlet (1978))

(5.6) $$\| v_h \| \leq \| \psi^\varepsilon - \rho_h \| \leq C \| \psi^\varepsilon \|_{2,2,\Omega} \cdot h$$

where $\rho_h = \sum_{i,j} \psi_{i,j}^\varepsilon w^{ij}(x)$. Thus the remaining part of this section will be devoted to the estimate of $\| \theta_h \|$.

5.2 Using (5.3), (5.4) and the relation

(5.7) $$\| \theta_h \| = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{|(\theta_h , v)|}{\| v \|} \leq \sup_{\substack{w_h \in H_{oh}^1(\Omega) \\ w_h \neq 0}} \frac{|(\theta_h , w_h)|}{\| w_h \|} ,$$

we have

$$\text{(5.8)} \qquad \| \theta_h \| \le \sup_{w_h \in H^1_{oh}(\Omega)} \frac{|(\psi^\varepsilon - \phi_h^\varepsilon, w_h)|}{\| w_h \|} .$$

Therefore we may compute $(\psi^\varepsilon - \phi_h^\varepsilon, w_h)$.

Multiplying w_h on both sides of (2.5) and integrating by parts, we have (see (5.1))

$$\text{(5.9)} \qquad (\psi^\varepsilon, w_h) = \int_{\Omega_\varepsilon} f\, w_h\, dx + \frac{1}{\varepsilon} \sum_{i,j} (\chi_h \psi^\varepsilon w_h)_{ij}\, h^2 - \frac{1}{\varepsilon} \int_{\Omega_1} \psi^\varepsilon w_h\, dx .$$

Similarly from (2.9) and (2.10),

$$\text{(5.10)} \qquad (\phi_h^\varepsilon, w_h) = \sum_{(i,j)\in \mathcal{R}_{oh}} (f\, w_h)_{ij}\, h^2 .$$

The subtraction of (5.10) from (5.9) gives

$$\text{(5.11)} \qquad (\psi^\varepsilon - \phi_h^\varepsilon, w_h) = \{\int_{\Omega_0} f\, w_h\, dx - \sum_{(i,j)\in \mathcal{R}_0} (f\, w_h)_{i,j}\, h^2\}$$

$$+ \{\frac{1}{\varepsilon} \sum_{(i,j)\in \mathcal{R}} (\chi_h \psi^\varepsilon w_h)_{i,j} h^2 - \frac{1}{\varepsilon} \int_{\Omega_1} \psi^\varepsilon w_h\, dx\}$$

$$= I + J .$$

Let $H_{oh}(\Omega)=\{w_h \in H_h^1(\Omega) \,|\, w_h|_{\partial\Omega} = 0\}$. Then, since $H_{oh}(\Omega)$ is a closed subspace in $H_0^1(\Omega)$, there exists one and only one element $\theta_h \in H_{oh}(\Omega)$ which minimize $\|\psi^\varepsilon - \phi_h^\varepsilon - w_h\|$ among $w_h \in H_{oh}^1(\Omega)$.

We denote

$$(5.3) \qquad v_h = \psi^\varepsilon - \phi_h^\varepsilon - \theta_h \,.$$

Obviously, v_h satisfies

$$(5.4) \qquad (v_h \,, w_h) = 0 \,, \qquad \forall w_h \in H_{oh}^1(\Omega)$$

and there holds

$$(5.5) \qquad \| \psi^\varepsilon - \phi_h^\varepsilon \|^2 = \| \theta_h \|^2 + \| v_h \|^2 \,.$$

Therefore we may evaluate both $\| \theta_h \|$ and $\| v_h \|$. It follows from (5.4) that for some C (see Ciarlet (1978))

$$(5.6) \qquad \| v_h \| \leq \| \psi^\varepsilon - \rho_h \| \leq C \| \psi^\varepsilon \|_{2,2,\Omega} \cdot h$$

where $\rho_h = \sum_{i,j} \psi_{i,j}^\varepsilon w^{ij}(x)$. Thus the remaining part of this section will be devoted to the estimate of $\| \theta_h \|$.

5.2 Using (5.3), (5.4) and the relation

$$(5.7) \qquad \| \theta_h \| = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{|(\theta_h \,, v)|}{\| v \|} \leq \sup_{\substack{w_h \in H_{oh}^1(\Omega) \\ w_h \neq 0}} \frac{|(\theta_h \,, w_h)|}{\| w_h \|} \,,$$

we have

$$\| \theta_h \| \leq \sup_{w_h \in H^1_{oh}(\Omega)} \frac{|(\psi^\varepsilon - \phi_h^\varepsilon, w_h)|}{\| w_h \|} . \tag{5.8}$$

Therefore we may compute $(\psi^\varepsilon - \phi_h^\varepsilon, w_h)$.

Multiplying w_h on both sides of (2.5) and integrating by parts, we have (see (5.1))

$$(\psi^\varepsilon, w_h) = \int_{\Omega_0} f\, w_h\, dx + \frac{1}{\varepsilon} \sum_{i,j} (\chi_h\, \psi^\varepsilon\, w_h)_{ij}\, h^2 - \frac{1}{\varepsilon} \int_{\Omega_1} \psi^\varepsilon\, w_h\, dx . \tag{5.9}$$

Similarly from (2.9) and (2.10),

$$(\phi_h^\varepsilon, w_h) = \sum_{(i,j)\in \Omega_{oh}} (f\, w_h)_{ij}\, h^2 . \tag{5.10}$$

The subtraction of (5.10) from (5.9) gives

$$\begin{aligned}(\psi^\varepsilon - \phi_h^\varepsilon, w_h) &= \{\int_{\Omega_0} f\, w_h\, dx - \sum_{(i,j)\in \Omega_0} (f\, w_h)_{i,j}\, h^2\} \\ &\quad + \{\frac{1}{\varepsilon} \sum_{(i,j)\in \Omega} (\chi_h\, \psi^\varepsilon\, w_h)_{i,j} h^2 - \frac{1}{\varepsilon}\int_{\Omega_1} \psi^\varepsilon\, w_h\, dx\} \\ &= I + J .\end{aligned} \tag{5.11}$$

Divide J into two parts in the following way:

$$(5.12)\qquad J = \{\frac{1}{\varepsilon}\sum_{i,j\in\hat{\Omega}_{1h}} (\psi^{\varepsilon} w_h)_{i,j}\, h^2 - \frac{1}{\varepsilon}\int_{\Omega_{1h}} \psi^{\varepsilon} w_h\, dx\}$$

$$+ \{\frac{1}{\varepsilon}\sum_{i,j\in\hat{D}} (\chi_h w_h)_{i,j}\, h^2 - \frac{1}{\varepsilon}\int_{D\cap\Omega_1} \psi^{\varepsilon} w_h\, dx\}$$

$$= J_1 + J_2 .$$

5.3 In order to estimate J_1 and J_2, we will prepare the following lemmas.

Lemma 5.1 (Quadrature formula on Ω)

$$(5.13)\qquad \left|\int_{\Omega} f\, w_h\, dx - \sum_{(i,j)\in\hat{\Omega}} (f\, w_h)_{i,j}\, h^2\right|$$

$$\leq ch^2 \sum_{(i,j)\in\hat{\Omega}} |f|_{2,\infty,\sigma_{ij}} \times |w_{hij}|\, h^2 \qquad (*)$$

where σ_{ij} is the support of w^{ij} (see (2.12)).

Proof: See Ciarlet (1978) (Error estimates for the quadrature formula on a regular mesh)

Lemma 5.2 (Quadrature formula on D)

(*) $|f|_{2,\infty,\sigma_{ij}}$ stands for the norm in $W^{2,\infty}(\sigma_{ij})$.

$$(5.14)\qquad \left|\int_D f\, w_h\, dx - \sum_{(i,j)\in\tilde{D}} (f\, w_h)_{ij}\, h^2\right|$$

$$\leq 3\sqrt{2}h \sum_{(i,j)\in\tilde{D}} |\nabla f|_{\infty,\sigma_{ij}} \cdot |w_{hij}|\, h^2$$

<u>Proof</u> Since $w_h(x) = \sum_{ij} w_{ij} w^{ij}(x)$,

$$(5.15)\qquad \left|\int_D f\, w_h\, dx - \sum_{(i,j)\in\tilde{D}} (f\, w_h)_{ij}\, h^2\right|$$

$$= \left|\sum_{(i,j)\in\tilde{D}} w_{ij} \int (f\, w^{ij} - f\, h^2)\, dx\right|$$

$$\leq \sum_{(i,j)\in\tilde{D}} |w_{ij}| \left|\int_{\sigma_{ij}} (f\, w^{ij} - f\, h^2)\, dx\right|.$$

There holds on σ_{ij}:

$$(5.16)\qquad |f(x) - f_{ij}| \leq |\nabla f|_{\infty,\sigma_{ij}} \cdot h\sqrt{2}$$

by means of the mean value theorem. We have

$$(5.17)\qquad \int_{\sigma_{ij}} w^{ij}\, dx = h^2$$

and

$$(5.18)\qquad \text{area of } \sigma_{ij} = 3h^2.$$

Summing up (5.15)-(5.18),

$$(5.19) \qquad \int_{\sigma_{ij}} (f\,w^{ij} - f\,h^2)\,dx \le |\nabla f|_{\infty,\sigma_{ij}} \cdot h/2 \cdot 3h^2.$$

Substituting (5.19) into (5.15), we conclude (5.14). ∎

<u>Lemma 5.3</u> (Approximation of an integral on a ship by a boundary integral). With the notation of the Figure 4,

$$(5.20) \qquad \left|\int_{D\cap\Omega_1} f\,dx - \int_D f\,\frac{\ell_1}{\ell_1+\ell_2}\,dx\right|$$

$$\le \int_\Gamma |\nabla f|_{\infty,\Gamma^\perp(s)} (\ell_1+\ell_2)^2(s)\,ds$$

<u>Proof</u> By the mean values theorem for integrals, we have

$$(5.21) \qquad \int_{D\cap\Omega_1} f\,dx = \int_\Gamma \left(\int_{\Gamma^\perp(s)} f(x)\,d\Gamma\right) d\Gamma$$

$$= \int_\Gamma (f(s) + \gamma(s))\,\ell_1(s)\,d\Gamma$$

$$(5.22) \qquad |\gamma(s)| \le |\nabla f|_{\infty,\Gamma^\perp(s)} \cdot (\ell_1 + \ell_2)(s).$$

By the same argument,

$$(5.23) \qquad \left|\int_\Gamma g\,d\Gamma - \int_D \frac{g}{\ell_1+\ell_2}\,dx\right| \le \int_\Gamma |\nabla g|_{\infty,\Gamma^\perp(s)} \cdot (\ell_1+\ell_2)(s)\,ds$$

where $(\ell_1+\ell_2)(x)$ is constant on $\Gamma^\perp(s)$ and equals to $(\ell_1+\ell_2)(s)$.

Hence we have (5.20). ∎

Lemma 5.4 (Discrete Poincaré inequality)

$$(5.24)\qquad \sum_{(i,j)\in\Omega} w_{ij}^2 h^2 \leq C\int_\Omega |\nabla w_h|^2 dx\ , \qquad \forall w_h\big|_{\partial\Omega} = 0$$

where C is the universal constant.

Proof

$$(5.25)\qquad w_{ij}^2 = \Big(\sum_{\ell=1}^{i} w_{\ell j} - w_{\ell-1j}\Big)^2 \leq i\sum_{\ell=1}^{i}\Big(\frac{w_{\ell j}-w_{\ell-1j}}{h}\Big)^2 h^2 \qquad (*)$$

From (5.25),

$$(5.26)\qquad \sum_{i,j=0}^{N} w_{ij}^2 h^2 \leq \sum_{ij} h^2 \{N\sum_{\ell=1}^{N}\Big(\frac{\phi_{\ell j}-\phi_{\ell-1j}}{h}\Big)^2 h^2\}$$

$$= \{\sum_{\ell j}\Big(\frac{\phi_{\ell j}-\phi_{\ell-1j}}{h}\Big)^2 h^2\}N^2h^2.$$

Therefore, we have

$$(5.27)\qquad \sum_{ij} w_{ij}^2 h^2 \leq C\sum_{\ell k}\{\Big(\frac{\phi_{\ell k}-\phi_{\ell-1k}}{h}\Big)^2 + \Big(\frac{\phi_{\ell k}-\phi_{\ell k-1}}{h}\Big)^2\}h^2.$$ ∎

Lemma 5.5 (Discrete trace theorem)

$$(5.28)\qquad \sum_{(i,j)\in\partial D} w_{ij}^2 h \leq C\int_\Omega |\nabla w_h|^2\, dx\ , \qquad \forall w_h\big|_{\partial\Omega} = 0$$

(*) Recall that $(\Sigma\ a)^N \leq N\,\Sigma\, a^N$

Proof Let us note that the number of points in $\tilde{D}$ is of order $\frac{1}{h}$. By (5.25)

$$(5.29) \qquad \sum_{(i,j)\in\tilde{D}} w_{ij}^2 h \le \sum_{(i,j)\in D} h\, N \sum_{\ell=1}^{N} h^2 \left(\frac{w_{\ell j}-w_{\ell-ij}}{h}\right)^2$$

$$\le C\int_{\Omega} |\nabla w_h|^2 dx \quad .$$

Hence we conclude (5.28). ∎

5.4

(i) Estimation of I: By Lemma 5.1,

$$(5.30) \qquad |I| \le Ch^2 \Big(\sum_{ij\in\tilde{\Omega}} \|f\|^2_{2,\infty,\sigma_{ij}} h^2\Big) \times \Big(\sum \cdot w_{ij}^2 h^2\Big)^{\frac{1}{2}}$$

(ii) Estimation of J_1: By Lemma 5.1,

$$(5.31) \qquad |J_1| \le C\,\frac{h^2}{\varepsilon}\Big(\sum_{(i,j)\in\tilde{\Omega}_{1h}} \|\psi^\varepsilon\|^2_{2,\infty,\sigma_{ij}} h^2\Big)^{\frac{1}{2}} \Big(\sum w_{ij}^2 h^2\Big)^{\frac{1}{2}} \quad .$$

(iii) Estimation of J_2: By Lemmas 5.2 and 5.3,

$$(5.32) \qquad |J_2| \le \left|\int_{D\cap\Omega_1} \psi^\varepsilon w_h\, dx - \int_D \psi^\varepsilon w_h \frac{\ell_1}{\ell_1+\ell_2}\, dx\right|$$

$$+ \left|\int_D \psi^\varepsilon w_h \frac{\ell_1}{\ell_1+\ell_2}\, dx - \sum_{(i,j)\in\tilde{D}} (\chi_h \psi^\varepsilon w_h)_{ij} h^2\right|$$

$$\leq \int_\Gamma |\nabla(\psi^\varepsilon w_h)|_{\infty,\Gamma^\perp} (\ell_1 + \ell_2)^2 \, d\Gamma$$

$$+ C h^2 \sum_{(i,j)\in\mathcal{D}} |w_{ij}| \left|\nabla(\psi^\varepsilon \frac{\ell_1}{\ell_1+\ell_2})\right|_{\infty,\sigma_{ij}} \cdot h ,$$

Here we see that the definition of χ_h plays an essential role in this estimate.

(iv) Estimation of J: Therefore

$$|J| \leq \frac{Ch^2}{\varepsilon}[(\sum_{i,j\in\mathcal{N}_{1h}} \|\psi^\varepsilon\|^2_{2,\infty,\sigma_{ij}} h^2)^{\frac{1}{2}} (\sum_{i,j\in\mathcal{N}} w_{ij}^2 h^2)^{\frac{1}{2}} \tag{5.33}$$

$$+ (\sum_{i,j\in\mathcal{D}} |\nabla(\psi^\varepsilon \frac{\ell_1}{\ell_1+\ell_2})|^2_{\infty,\sigma_{ij}} h)^{\frac{1}{2}} (\sum_{i,j\in\mathcal{D}} w_{ij}^2 h)^{\frac{1}{2}}$$

$$+ (\int_\Gamma |\nabla\psi^\varepsilon|^2_{\infty,\Gamma^\perp} \cdot (\frac{\ell_1+\ell_2}{h})^4 d\Gamma)^{\frac{1}{2}} \cdot (\int_\Gamma |w_h|^2_{\infty,\Gamma^\perp} d\Gamma)^{\frac{1}{2}}$$

$$+ (\int_\Gamma |\psi^\varepsilon|^2_{\infty,\Gamma^\perp} \cdot (\frac{\ell_1+\ell_2}{h})^4 d\Gamma)^{\frac{1}{2}} \cdot (\int_\Gamma |\nabla w_h|^2_{\infty,\Gamma^\perp} d\Gamma)^{\frac{1}{2}}].$$

By lemma 5.4 and 5.5, the sums of w_{ij} in (5.33) are replaced by $(\int_\Omega |\nabla w_h|^2 dx)^{1/2}$.

Now $|w_h|_{\infty,\Gamma_{ij}} = w_{k\ell}$, $(kh,\ell h) \in \sigma_{ij}$ and $|\nabla w_h|_{\infty,\Gamma} \leq 2|w_h|_{\infty,\Gamma^\perp}/h$. Therefore, for some curve $\tilde{\Gamma}$ near Γ

$$(5.34) \qquad \int_\Gamma |\nabla w_h|^2_{\infty,\Gamma^\perp} d\Gamma \le \frac{C_1}{h^2}\int_\Gamma |w_h|^2_{\infty,\Gamma^\perp} d\Gamma \le \frac{C_2}{h^2} \sum_{(i,j)\in\tilde{\Gamma}} w_{ij}^2 h$$

$$\le \frac{C_3}{h^2}\int_\Omega |\nabla w_h|^2 \, dx \qquad \text{(by Lemma 5.5).}$$

Thus, summing up the above results, we have

$$(5.35) \qquad \| \theta_h \| \le (|I| + |J|)/(\int_\Omega |\nabla w_h|^2 dx)^{\frac{1}{2}}$$

$$\le Ch^2 \Big(\sum_{(i,j)\in\tilde{\Omega}} \| f \|^2_{2,\infty,\sigma_{ij}} h^2 \Big)$$

$$+ \frac{Ch^2}{\varepsilon} \Big[\Big(\sum_{(i,j)\in\tilde{\Omega}_{1h}} \| \psi^\varepsilon \|^2_{2,\infty,\sigma_{ij}} h^2 \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{(i,j)\in\tilde{D}} \Big| \nabla \Big(\psi^\varepsilon \frac{\ell_1}{\ell_1+\ell_2}\Big) \Big|^2_{\infty,\sigma_{ij}} h \Big)^{\frac{1}{2}}$$

$$+ \Big(\int_\Gamma |\nabla \psi^\varepsilon|^2_{\infty,\Gamma^\perp} \Big(\frac{\ell_1+\ell_2}{h}\Big)^4 d\Gamma \Big)^{\frac{1}{2}}$$

$$+ \frac{\sqrt{\varepsilon}}{h} \Big(\int_\Gamma \Big| \frac{\psi^\varepsilon}{\sqrt{\varepsilon}} \Big|^2_{\infty,\Gamma^\perp} \Big(\frac{\ell_1+\ell_2}{h}\Big)^4 d\Gamma \Big)^{\frac{1}{2}} \Big]$$

5.5 We are now in the final step:

$$(5.36)\qquad \| \psi^{\varepsilon} - \phi_h^{\varepsilon} \| = \| v_h \| + \| \theta_h \|$$

$$\leq \| \psi^{\varepsilon} \|_{2,2,\Omega} \cdot h + Ch^2 \Big(\sum_{(i,j)\in \tilde{\Omega}} \| f \|^2_{2,\infty,\sigma_{ij}} h^2 \Big)^{\frac{1}{2}}$$

$$+ \frac{Ch^2}{\varepsilon} \cdot D(\psi^{\varepsilon}) .$$

$$(5.37)\qquad D(\psi^{\varepsilon}) = \Big(\sum_{(i,j)\in \tilde{\Omega}_{1h}} \| \psi^{\varepsilon} \|^2_{2,\infty,\sigma_{ij}} \cdot h^2 \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{(i,j)\in \tilde{\mathcal{B}}} \Big| \nabla \big(\psi^{\varepsilon} \frac{\ell_1}{\ell_1 + \ell_2} \big) \Big|^2_{\infty,\sigma_{ij}} h \Big)^{\frac{1}{2}}$$

$$+ \Big(\int_{\Gamma} |\nabla \psi^{\varepsilon}|_{\infty,\Gamma^{\perp}} \big(\frac{\ell_1 + \ell_2}{h} \big)^4 \, d\Gamma \Big)^{\frac{1}{2}}$$

$$+ \frac{\sqrt{\varepsilon}}{h} \Big(\int_{\Gamma} \Big| \frac{\psi^{\varepsilon}}{\sqrt{\varepsilon}} \Big|_{\infty,\Gamma^{\perp}} \big(\frac{\ell_1 + \ell_2}{h} \big)^4 \, d\Gamma \Big)^{\frac{1}{2}} .$$

By using the regularity results in Theorem 3,

$$(5.38)\qquad \Big\| \psi^{\varepsilon} - \phi_h^{\varepsilon} \Big\|_{1,\Omega} \leq C \Big(\frac{h}{\sqrt{\varepsilon}} + \frac{h^2}{\varepsilon\sqrt{\varepsilon}} \Big) .$$

6. The Algorithm for Solving Free Boundary Problems

6.1 For the sake of simplicity, we have tried our method on a simple geometry where the upper boundary of an open rectangle

is free (see Fig. 5). Consider the following problem; find ϕ and γ such that

(6.1) $$-\Delta\phi = f \quad \text{in } \Omega_\gamma$$

(6.2) $$\phi = 0\ , \quad \frac{\partial\psi}{\partial n} = g \quad \text{on } \gamma$$

(6.3) $$\phi(0\ , x_2) = \phi(1\ , x_2) \quad (x_2 > 0)$$

(6.4) $$\phi(x_1\ , 0) = 0 \quad (0 < x_1 < 1).$$

We assume that the free boundary γ is represented by the equation $\eta = \eta(x_1)$ $(0 < x_1 < 1)$ and satisfy $0 < \eta(x_1) < 1$ $(0 < x_1 < 1)$.

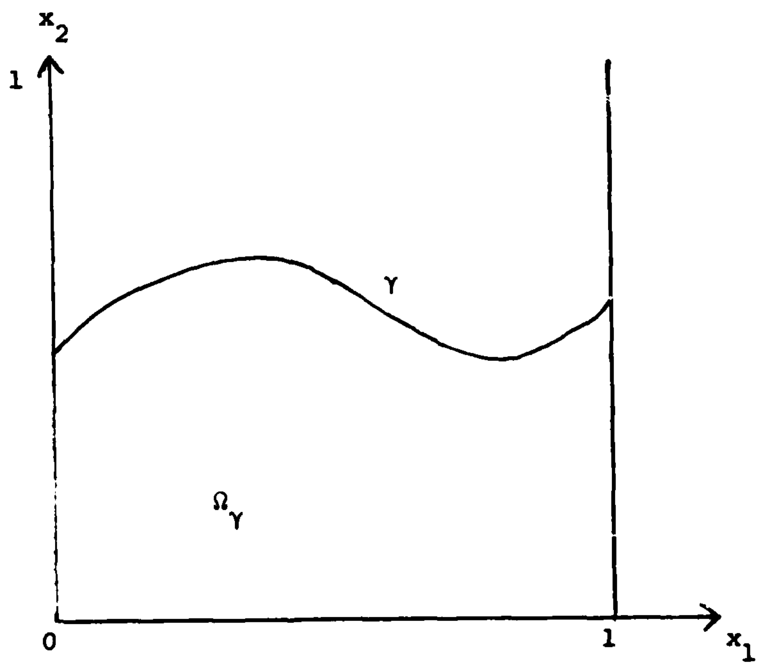

Figure 5

Then the discrete problem corresponding to the penalized state equation is given by:

$$(6.5) \qquad -(\Delta_h \phi^{\varepsilon})_{i,j} + \frac{1}{\varepsilon}(\chi_h \phi^{\varepsilon})_{i,j} = ((1-\chi_h)f)_{ij} \qquad \text{in } \tilde{\mathcal{C}} ,$$

$$(6.6) \qquad (\phi^{\varepsilon})_{0,j} = (\phi^{\varepsilon})_{N,j} \qquad (0 \leq j \leq N)$$

$$(6.7) \qquad (\phi^{\varepsilon})_{i,0} = (\phi^{\varepsilon})_{i,N} \qquad (1 \leq i \leq N-1).$$

As the boundary γ $(y = \eta(x))$ defining χ_h, we may choose the linear interpolation $y = \eta_h(x)$ $(0 < x < 1)$ of $\eta^i = \eta(ih)$ $(0 \leq i \leq N)$. We take χ_h to be the same as (2.16).

6.2 Define

$$(6.8) \qquad g_h(x) = -\frac{1}{\sqrt{\varepsilon}} \phi_h^{\varepsilon}(x, \eta_h(x))$$

$$(6.9) \qquad g^i = g(ih, \eta^i).$$

Then we introduce the discrete optimization problem

$$(6.10) \qquad \min \{E_h(\eta_h) = h \sum_{i=0}^{N-1} |g_h(ih) - g^i|^2\} .$$

6.3 Owing to the practical importance of $\frac{\partial E_h}{\partial \eta_h}$, we shall discuss its computation. For the sake of simplicity, we take into account only the variation of y-direction with respect to differentiation of E_h. Define

$$(6.11) \qquad \underset{\sim}{\eta} = \{\eta^i\} \qquad (\eta^i = \eta_h(ih))$$

$$(6.12) \qquad \underset{\sim}{\eta}' = \{\eta^i \in \delta\eta^k \cdot \delta_{i,k}\} \ .$$

Let $\delta\eta^k$ be the small deviation of the k th component of $\underset{\sim}{\eta}$. Then we have

$$(6.13) \qquad \frac{\partial E_h}{\partial \eta^k} = 2h \sum_{i=0}^{N-1} (g_h(ih) - g^i) \frac{\partial g_h(ih)}{\partial \eta^k} \ .$$

Now

$$(6.14) \qquad g_h(ih) = \frac{1}{\sqrt{\varepsilon}} \phi_h^{\varepsilon}(ih, \eta_h(ih)) = \frac{1}{\sqrt{\varepsilon}} [\phi^{\varepsilon}_{i,J^i}\theta^i + \phi^{\varepsilon}_{i,J^i}(1 - \theta^i)]$$

where θ^i is defined by

$$(6.15) \qquad \eta^i = h[\theta^i(J^i - 1) + (1 - \theta^i)J^i]$$

Therefore

$$(6.16) \qquad \frac{\partial g_h}{\partial \eta^k}(ih) = \frac{1}{\sqrt{\varepsilon}}[\frac{\partial}{\partial \eta^k}(\phi^{\varepsilon}_{i,J^i-1})\theta^i + \frac{\partial}{\partial \eta^k}(\phi^{\varepsilon}_{i,J^i})(1 - \theta^i)$$

$$+ \frac{\partial \theta^i}{\partial \eta^k}(\phi^{\varepsilon}_{i,J^i-1} - \phi^{\varepsilon}_{i,J^i})] \ ,$$

but from (6.15)

$$(6.17) \qquad \delta_{i,k} = -h\frac{\partial \theta^i}{\partial \eta^k} \ .$$

To evaluate $\phi'_{ij} = \partial\phi^{\varepsilon}_{ij}/\partial\eta^k$ we differentiate (6.5) with respect to η^k:

$$(6.18)\qquad -(\Delta_h\phi'_{ij} + \frac{1}{\varepsilon}(\chi_h\phi')_{ij} = -\frac{\partial}{\partial\eta^k}(\chi_h f)_{ij} - \frac{1}{\varepsilon}(\frac{\partial\chi_h}{\partial\eta^k}\phi^\varepsilon)_{ij}$$

(plus some boundary conditions as in (6.6)-(6.7))

Let $\{p^\varepsilon_{ij}\}$ be the solution of

$$(6.19)\qquad -(\Delta_h p^\varepsilon)_{ij} + \frac{1}{\varepsilon}(\chi_h p^\varepsilon)_{ij}$$

$$= (g_h(ih) - g^i)(\theta^i\delta_{j,J^i-1} + (1-\theta^i)\delta_{j,J^i})$$

$$p^\varepsilon_{0,j} = p^\varepsilon_{N,j} \qquad 1 \leq j \leq N-1$$

$$p^\varepsilon_{i,0} = p^\varepsilon_{L,N} = 0 \qquad 0 \leq l \leq N$$

and multiply (6.19) by ϕ'_{ij} and sum over all i,j; this brings

$$(6.20)\qquad \sum_i (g_h(ih) - g^i)(\theta^i\phi'_{i,J^i-1} + (1-\theta^i)\phi'_{i,J^i})$$

$$= -\sum_{ij} p^\varepsilon_{ij}(\frac{\partial}{\partial\eta^k}(\chi_h f)_{ij} + \frac{1}{\varepsilon}(\frac{\partial\chi_h}{\partial\eta^k}\phi^\varepsilon)_{ij})$$

Therefore

$$(6.21)\qquad \frac{\partial E_h}{\partial\eta^k} = -2h\sum_{i=0}^{N-1}(g_h(ih) - g^i)\frac{\partial g^i}{\partial\eta^k}$$

$$-\frac{2h}{\sqrt{\varepsilon}}[\sum_{ij} p^\varepsilon_{ij}(\frac{\partial}{\partial\eta^k}(\chi_h f)_{ij} + \frac{1}{\varepsilon}(\frac{\partial\chi_h}{\partial\eta^k}\phi^\varepsilon)_{ij})]$$

$$-\frac{2}{\sqrt{\varepsilon}}(g_h(kh) - g^k)(\phi^\varepsilon_{k,J^k-1} - \phi^\varepsilon_{k,J^k}) \equiv M^k_h$$

(6.11) $$\underset{\sim}{\eta} = \{\eta^i\} \qquad (\eta^i = \eta_h(ih))$$

(6.12) $$\underset{\sim}{\eta}' = \{\eta^i \in \delta\eta^k \cdot \delta_{i,k}\} \ .$$

Let $\delta\eta^k$ be the small deviation of the k th component of $\underset{\sim}{\eta}$. Then we have

(6.13) $$\frac{\partial E_h}{\partial \eta^k} = 2h \sum_{i=0}^{N-1} (g_h(ih) - g^i) \frac{\partial g_h(ih)}{\partial \eta^k} \ .$$

Now

(6.14) $$g_h(ih) = \frac{1}{\sqrt{\varepsilon}} \phi_h^\varepsilon(ih , \eta_h(ih) = \frac{1}{\sqrt{\varepsilon}} [\phi^\varepsilon_{i,J^i} \theta^i + \phi^\varepsilon_{i,J^i}(1 - \theta^i)]$$

where θ^i is defined by

(6.15) $$\eta^i = h[\theta^i(J^i - 1) + (1 - \theta^i)J^i]$$

Therefore

(6.16) $$\frac{\partial g_h}{\partial \eta^k}(ih) = \frac{1}{\sqrt{\varepsilon}}[\frac{\partial}{\partial \eta^k}(\phi^\varepsilon_{i,J^i-1})\theta^i + \frac{\partial}{\partial \eta^k}(\phi^\varepsilon_{i,J^i})(1 - \theta^i)$$
$$+ \frac{\partial \theta^i}{\partial \eta^k}(\phi^\varepsilon_{i,J^i-1} - \phi^\varepsilon_{i,J^i})] \ ,$$

but from (6.15)

(6.17) $$\delta_{i,k} = - h \frac{\partial \theta^i}{\partial \eta^k} \ .$$

To evaluate $\phi'_{ij} = \partial\phi^\varepsilon_{ij}/\partial\eta^k$ we differentiate (6.5) with respect to η^k:

$$(6.18)\qquad -(\Delta_h\phi'_{ij} + \frac{1}{\varepsilon}(\chi_h\phi')_{ij} = -\frac{\partial}{\partial\eta^k}(\chi_h f)_{ij} - \frac{1}{\varepsilon}(\frac{\partial\chi_h}{\partial\eta^k}\phi^\varepsilon)_{ij}$$

(plus some boundary conditions as in (6.6)-(6.7))

Let $\{p^\varepsilon_{ij}\}$ be the solution of

$$(6.19)\qquad -(\Delta_h p^\varepsilon)_{ij} + \frac{1}{\varepsilon}(\chi_h p^\varepsilon)_{ij}$$

$$= (g_h(ih) - g^i)(\theta^i\delta_{j,J^i-1} + (1-\theta^i)\delta_{j,J^i})$$

$$p^\varepsilon_{0,j} = p^\varepsilon_{N,j} \qquad 1 \le j \le N-1$$

$$p^\varepsilon_{i,0} = p^\varepsilon_{L,N} = 0 \qquad 0 \le l \le N$$

and multiply (6.19) by ϕ'_{ij} and sum over all i,j; this brings

$$(6.20)\qquad \sum_i (g_h(ih) - g^i)(\theta^i\phi'_{i,J^i-1} + (1-\theta^i)\phi'_{i,J^i})$$

$$= -\sum_{ij} p^\varepsilon_{ij}(\frac{\partial}{\partial\eta^k}(\chi_h f)_{ij} + \frac{1}{\varepsilon}(\frac{\partial\chi_h}{\partial\eta^k}\phi^\varepsilon)_{ij})$$

Therefore

$$(6.21)\qquad \frac{\partial E_h}{\partial\eta^k} = -2h\sum_{i=0}^{N-1}(g_h(ih) - g^i)\frac{\partial g^i}{\partial\eta^k}$$

$$-\frac{2h}{\sqrt{\varepsilon}}[\sum_{ij} p^\varepsilon_{ij}(\frac{\partial}{\partial\eta^k}(\chi_h f)_{ij} + \frac{1}{\varepsilon}(\frac{\partial\chi_h}{\partial\eta^k}\phi^\varepsilon)_{ij})]$$

$$-\frac{2}{\sqrt{\varepsilon}}(g_h(kh) - g^k)(\phi^\varepsilon_{k,J^k-1} - \phi^\varepsilon_{k,J^k}) \equiv M^k_h$$

(6.22) Take $\delta\eta^k = -\lambda M_h^k \qquad (0 < \lambda << 1)$.

Then we have

(6.23) $$E_h(\underset{\sim}{\eta}') - E_h(\underset{\sim}{\eta}) = -\lambda(M_h^k)^2 < 0.$$

6.4 Therefore let us consider the following algorithm for solving (6.10):

<u>step 1</u> Choose $\lambda > 0$ small, choose the initial guess of the free boundary $\underset{\sim}{\eta_h} = \{\eta^i\}$.

<u>step 2</u> Determine χ_h by $\underset{\sim}{\eta_h}$

<u>step 3</u> Compute ϕ_h^ε by solving (6.5)-(6.7).

<u>step 4</u> Compute p_h^ε by solving (6.14)-(6.16).

<u>step 5</u> Set

$$\delta\eta^k = -\lambda M_h^k \qquad (0 \leq k \leq N-1).$$

and

$$\eta_h' = \{\eta^i + \delta\eta^k \cdot \delta_{ki}\}$$

replace η_h by η_h' and go back to step 2.

6.5 As an example, we shall deal with the problem (6.1)-(6.4). Which has a solution $\{y = \eta(x) = a + b\sin 2\pi x,\ \phi(x,y) = cy(\eta(x) - y)\}$. Naturally, the data is defined

(6.24) $$f = C\cdot\left(2 - y\frac{d^2\eta}{dx^2}\right) = C(2 + 4\pi^2 y \sin^2 \pi x)$$

(6.25) $$g = -C\eta\cdot\left(1 + \left(\frac{d\eta}{dx}\right)^2\right)^{\frac{1}{2}} = -C\sin 2\pi x\,(1 + 4\pi^2 \sin^2 2\pi)^{\frac{1}{2}}$$

6.5 By using the algorithm mentioned above, we have obtained several numerical results, which will be reported in following papers.

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We study a numerical method for solving free boundary problems of elliptic type. Usually these problems are prescribed with two boundary conditions on the free boundary. One of them is the Dirichlet condition and the other is the Neumann condition. Our method is to transform the original problem to an optimization problem. The state equation is approximated by an equation with a penalty term, in which the Dirichlet condition on the free boundary is approximately satisfied. The outward normal derivative included in the Neumann condition through the free boundary is calculated by using the

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ABSTRACT (continued)

asymptotic behavior of the solution of the penalized state equation. Here we present a method to solve this penalized optimization problem. Also the error estimate of the discretized state equation by the finite difference method is given.

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